Integrable hierarchies of nonlinear difference-difference equations and symmetries

2001 J. Phys. A: Math. Gen. 34 10357

(http://iopscience.iop.org/0305-4470/34/48/302)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 10357–10368 PII: S0305-4470(01)23939-0

Integrable hierarchies of nonlinear difference–difference equations and symmetries

D Levi[1] **and L Martina**[2]

[1] Dipartimento di Fisica 'E Amaldi', Universitá degli Studi Roma Tre and Sezione INFN, Roma Tre, Via della Vasca Navale 84, 00146 Roma, Italy
[2] Dipartimento di Fisica, Universitá di Lecce and Sezione INFN di Lecce, Lecce, Italy

Received 17 April 2001, in final form 17 August 2001
Published 23 November 2001
Online at stacks.iop.org/JPhysA/34/10357

Abstract

We construct the hierarchy of nonlinear difference–difference equations associated with the discrete Schrödinger spectral problem. As examples of equations contained in this hierarchy we obtain the discrete-time Toda and Volterra lattice equations. In the case of the time-discrete Toda lattice, we construct its Lie point and generalized symmetries. Finally, we present its Bäcklund transformations and relate it to the already constructed symmetries.

PACS numbers: 02.30.Ik, 02.30.Ks, 05.45.Yv

1. Introduction

Nonlinear integrable differential or differential–difference equations appear in the form of hierarchies of equations [1–3], all characterized by a common spectral problem and by the existence of a recursion operator. Equations belonging to the same hierarchy share many properties connected to the presence of a common spectral problem: among them are the presence of Bäcklund transformations and symmetries.

Many integrable nonlinear discrete–discrete equations have been considered in the literature [4–10], but up to now no one has considered hierarchies of nonlinear discrete–discrete equations. Here we shall apply a technique previously used for obtaining hierarchies of differential or differential–difference equations. More specifically we construct the recursion operator that will produce hierarchies of difference–difference equations.

We shall determine the symmetries of these discrete equations, making use of their integrability. First we shall consider the Lie point symmetries [11–15], acting on the lattice and on the equations, using an adapted evolutionary formalism. The peculiar structure of the lattice will introduce severe restrictions, as will be seen in the examples. Next, we shall consider generalized symmetries, in the evolutionary formalism, by exploiting the existence of infinitely many differential–difference equations in involution in the spectral parameter space. In particular we shall show the peculiar role of certain nonlinear differential–difference equations as generalized symmetries of the nonlinear difference–difference equations. For the sake of concreteness we shall focus on just one equation of the hierarchy, the discrete-time Toda lattice. The results presented can be easily extended to all the equations. In this case, even though our equations are completely discrete and Bäcklund transformations are given by difference equations, infinitesimal symmetries are given by differential–difference equations, as in the case of differential–difference equations.

In section 2, starting from the discrete Schrödinger spectral problem we construct the associated hierarchy of nonlinear difference–difference equations. As examples, we consider the discrete-time Toda lattice and the Volterra equations. For completeness we present both their continuum time limit and the discrete evolution of the reflection and transmission coefficients.

Section 3 is devoted to the calculation of the Lie point symmetries of the discrete-time Toda lattice using a recently developed procedure [16]. In section 4 we construct the isospectral and non-isospectral symmetries starting from the integrability properties of the equation.

In section 5 we construct the Bäcklund transformations for the hierarchy and show their relation to the group transformations obtained by integrating the infinitesimal symmetries constructed in section 4. Section 6 is devoted to some concluding remarks.

2. Construction of the discrete-time Toda lattice hierarchy

We start from the discrete Schrödinger spectral problem

$$\psi_{n-1,m} + a_{n,m}\psi_{n+1,m} + b_{n,m}\psi_{n,m} \equiv L_{n,m}\psi_{n,m} = \lambda\psi_{n,m} \tag{1}$$

where $a_{n,m}$ and $b_{n,m}$, for any m, reduce to unity and zero respectively , as n goes to infinity. In equation (1) λ is an m-independent spectral parameter. An integrable nonlinear difference–difference equation can be written in operator form as

$$L_{n,m+1} - L_{n,m} = L_{n,m+1}M_{n,m} - M_{n,m}L_{n,m} \tag{2}$$

in terms of the operator $M_{n,m}$ which governs the discrete 'time' evolution of the wavefunction $\psi_{n,m}$ of equation (1)

$$\psi_{n,m+1} = \psi_{n,m} - M_{n,m}\psi_{n,m}. \tag{3}$$

Let us notice that for the operator $L_{n,m}$ given by equation (1) we can write

$$L_{n,m+1} - L_{n,m} = (a_{n,m+1} - a_{n,m})E_n^+ + b_{n,m+1} - b_{n,m} \tag{4}$$

where E_n^+ is the shift operator in the n-variable such that $E_n^+ f_{n,m} = f_{n+1,m}$ for any function $f_{n,m}$.

We use the by now standard Lax technique [3], in a similar way to the construction of the Toda lattice hierarchy. We construct a hierarchy of nonlinear discrete–discrete equations by requiring that an operator $M_{n,m}$ and two scalar functions $U_{n,m}$ and $V_{n,m}$ satisfy

$$L_{n,m+1}M_{n,m} - M_{n,m}L_{n,m} = U_{n,m}E_n^+ + V_{n,m}. \tag{5}$$

We then construct new functions $\tilde{U}_{n,m}$ and $\tilde{V}_{n,m}$ and a new operator $\tilde{M}_{n,m}$, using the following formulae:

$$L_{n,m+1}\tilde{M}_{n,m} - \tilde{M}_{n,m}L_{n,m} = \tilde{U}_{n,m}E_n^+ + \tilde{V}_{n,m} \tag{6}$$

$$\tilde{M}_{n,m} = L_{n,m+1}M_{n,m} + F_{n,m}E_n^+ + G_{n,m} \tag{7}$$

where $F_{n,m}$ and $G_{n,m}$ are two scalar functions. Imposing the compatibility condition of equations (1), (5)–(7) we obtain the following hierarchy of equations:

$$\begin{pmatrix} a_{n,m+1} - a_{n,m} \\ b_{n,m+1} - b_{n,m} \end{pmatrix} = f_m^1(\mathcal{L}_{n,m}) \begin{pmatrix} (b_{n,m+1} - b_{n+1,m})\frac{\pi_{n,m+1}}{\pi_{n+1,m}} \\ \frac{\pi_{n-1,m+1}}{\pi_{n,m}} - \frac{\pi_{n,m+1}}{\pi_{n+1,m}} \end{pmatrix} + f_m^2(\mathcal{L}_{n,m}) \begin{pmatrix} a_{n,m+1} - a_{n,m} \\ b_{n,m+1} - b_{n,m} \end{pmatrix}. \tag{8}$$

Here f_m^1 and f_m^2 are entire functions of their argument and $\mathcal{L}$ is the recursion operator of the hierarchy, obtained from equations (1) and (5)–(7) and given by

$$\mathcal{L}_{n,m}\begin{pmatrix} p_{n,m} \\ q_{n,m} \end{pmatrix} = \begin{pmatrix} a_{n,m+1}S_{n+2,m} - a_{n,m}S_{n,m} \\ p_{n-1,m} + \Sigma_{n-1,m}\frac{\pi_{n-1,m+1}}{\pi_{n,m}} - \Sigma_{n,m}\frac{\pi_{n,m+1}}{\pi_{n+1,m}} \end{pmatrix} + \begin{pmatrix} b_{n,m+1}p_{n,m} + (b_{n,m+1} - b_{n+1,m})\Sigma_{n,m}\frac{\pi_{n,m+1}}{\pi_{n+1,m}} \\ +b_{n,m+1}q_{n,m} + (b_{n,m+1} - b_{n,m})S_{n,m} \end{pmatrix}. \tag{9}$$

The starting points

$$\begin{pmatrix} (b_{n,m+1} - b_{n+1,m})\frac{\pi_{n,m+1}}{\pi_{n+1,m}} \\ \frac{\pi_{n-1,m+1}}{\pi_{n,m}} - \frac{\pi_{n,m+1}}{\pi_{n+1,m}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{n,m+1} - a_{n,m} \\ b_{n,m+1} - b_{n,m} \end{pmatrix}$$

are obtained as coefficients of the integration constants for the functions $F_{n,m}$ and $G_{n,m}$. The function $\pi_{n,m}$ is given by

$$\pi_{n,m} = \prod_{j=n}^{\infty} a_{j,m} \tag{10}$$

while $S_{n,m}$ and $\Sigma_{n,m}$ are defined as the *bounded solutions* of the equations

$$\begin{aligned} S_{n+1,m} - S_{n,m} &= q_{n,m} \\ \Sigma_{n+1,m} - \Sigma_{n,m} &= -p_{n+1,m}\frac{\pi_{n+2,m}}{\pi_{n+1,m+1}}. \end{aligned} \tag{11}$$

The boundedness of the solutions of equations (11) was not required in the literature [2, 3] but it is necessary to obtain a hierarchy of nonlinear difference–difference equations with well defined evolution of the spectra.

Let us define the reflection and transmission coefficients $R_m(z)$ and $T_m(z)$ in terms of the asymptotic behaviour of the function $\psi_{n,m}$

$$\begin{aligned} \lim_{n\to\infty}\psi_{n,m}(z) &= \phi_m(z^{-n} + R_m z^m) \\ \lim_{n\to-\infty}\psi_{n,m}(z) &= \phi_m T_m z^{-n} \end{aligned} \tag{12}$$

where ϕ_m is an appropriate normalization function depending just on m. In the case of a generic equation of the discrete Toda lattice hierarchy (8) the discrete evolution of the reflection coefficient turns out to be

$$R_{m+1} = \frac{1 - f_m^2(\lambda) - zf_m^1(\lambda)}{1 - f_m^2(\lambda) - \frac{f_m^1(\lambda)}{z}}R_m. \tag{13}$$

The transmission coefficient T_m does not evolve in m.

Let us notice that, as opposed to the usual case of hierarchies of partial differential or differential–difference equations, the recursion operator (9) depends on both the functions $(a_{n,m}, b_{n,m})$ and $(a_{n,m+1}, b_{n,m+1})$. Thus, in order to write the nonlinear difference–difference as an evolution equation in which we explicate the fields at the time $m + 1$ in terms of those at the time m, we must write down explicitly the complete system of equations and then solve for the fields at the time $m + 1$. It is not guaranteed that this can always be carried out since the equation can represent an implicit evolution in the discrete time.

Let us write down the simplest members of the hierarchy (8).

2.1. The discrete Toda lattice

Choosing $f_m^2 = 0$ and $f_m^1 = \alpha$ in equation (8) we obtain

$$a_{n,m+1} - a_{n,m} = \alpha(b_{n,m+1} - b_{n+1,m})\frac{\pi_{n,m+1}}{\pi_{n+1,m}} \tag{14}$$

$$b_{n,m+1} - b_{n,m} = \alpha\left(\frac{\pi_{n-1,m+1}}{\pi_{n,m}} - \frac{\pi_{n,m+1}}{\pi_{n+1,m}}\right). \tag{15}$$

Solving equations (14) and (15) for $b_{n+1,m} - b_{n,m}$ and taking into account the boundary conditions for the fields $a_{n,m}$ and $b_{n,m}$, we obtain

$$b_{n,m} = \alpha + \frac{1}{\alpha} - \alpha\frac{\pi_{n-1,m+1}}{\pi_{n,m}} - \frac{\pi_{n,m}}{\alpha\pi_{n,m+1}}. \tag{16}$$

Substituting equation (16) into (14) we obtain a single equation of higher order for the field $\pi_{n,m}$:

$$\Delta_{\text{Toda}} = \pi_{n-1,m+2} - \frac{1}{\alpha^2}\pi_{n,m} - \pi_{n,m+1}^2\left(\frac{1}{\pi_{n+1,m}} - \frac{1}{\alpha^2\pi_{n,m+2}}\right) = 0 \tag{17}$$

which, for $\pi_{n,m} = \mathrm{e}^{u_{n,m}}$, reads

$$\mathrm{e}^{u_{n,m}-u_{n,m+1}} - \mathrm{e}^{u_{n,m+1}-u_{n,m+2}} = \alpha^2(\mathrm{e}^{u_{n-1,m+2}-u_{n,m+1}} - \mathrm{e}^{u_{n,m+1}-u_{n+1,m}}) \tag{18}$$

i.e. the well known discrete-time Toda lattice equation [6]. On the left-hand side of equation (18) we can easily obtain the second difference of the function $u_{n,m}$ with respect to the discrete-time m. Thus, defining

$$t = m\sigma \qquad v_n(t) = u_{n,m} \qquad \alpha = \sigma^2 \tag{19}$$

we find that equation (18) reduces to the continuous-time Toda lattice equation

$$\ddot{v}_n = \mathrm{e}^{v_{n-1}-v_n} - \mathrm{e}^{v_n-v_{n+1}} + \mathrm{O}(\sigma). \tag{20}$$

Equation (18) has the following Lax pair:

$$\psi_{n-1,m} + \left(\alpha + \frac{1}{\alpha} - \alpha\mathrm{e}^{u_{n-1,m+1}-u_{n,m}} - \frac{\mathrm{e}^{u_{n,m}-u_{n,n+1}}}{\alpha}\right)\psi_{n,m} + \mathrm{e}^{u_{n,m}-u_{n+1,m}}\psi_{n+1,m} = \lambda\psi_{n,m} \tag{21}$$

$$\psi_{n,m+1} = \psi_{n,m} - \alpha\mathrm{e}^{u_{n,m+1}-u_{n+1,m}}\psi_{n+1,m}. \tag{22}$$

From equation (13) we obtain the evolution of the reflection coeffcient R_m and T_m:

$$R_{m+1} = \frac{1-\alpha z}{1-\frac{\alpha}{z}}R_m. \tag{23}$$

2.2. *The discrete non-isospectral Toda lattice*

A new discrete-time Toda-like equation can be obtained by imposing non-isospectral deformations on the spectral problem (1). This is the case when we allow λ to depend on m, i.e. $\lambda = \lambda_m$. In such a case $a_{n,m}$ and $b_{n,m}$ are no longer bounded asymptotically in n and we have not been able to construct a hierarchy of equations. However, we can obtain a nonlinear equation by assuming that λ_m evolves in m according to the equation

$$\lambda_{m+1} = \alpha_0 + \alpha_1\lambda_m \tag{24}$$

whose solution is given by

$$\lambda_m = \lambda_0\alpha_1^m + \alpha_0\frac{\alpha_1^m - 1}{\alpha_1 - 1}. \tag{25}$$

This is the only non-isospectral deformation compatible. We obtain the following difference–difference equation:

$$\frac{\alpha_0}{\alpha_1} + \frac{K_m^0}{K_m^1}\frac{1}{\alpha_1^{2n}}\frac{\pi_{n,m}}{\pi_{n,m+1}} - \frac{K_{m+1}^1}{K_{m+1}^0}\alpha_1^{2n-3}\frac{\pi_{n-1,m+2}}{\pi_{n,m+1}} = \frac{K_{m+1}^0}{K_{m+1}^1}\frac{1}{\alpha_1^{2n+1}}\frac{\pi_{n,m+1}}{\pi_{n,m+2}} - \frac{K_m^1}{K_m^0}\alpha_1^{2n}\frac{\pi_{n,m+1}}{\pi_{n+1,m}} \tag{26}$$

where K_m^1 and K_m^0 are arbitrary discrete-time dependent coefficients. They enter the discrete-time evolution of the linear spectral operator (1):

$$\psi_{n,m+1} = \frac{K_m^0}{\alpha_1^n}\psi_{n,m} + \alpha_1^n K_m^1\frac{\pi_{n,m+1}}{\pi_{n+1,m}}\psi_{n+1,m}. \tag{27}$$

As in the case of the simpler non-isospectral deformations of the Toda lattice [19], equation (26) is transformed into equation (17) for $\tilde{a}_{n,m}$ and $\tilde{b}_{n,m}$ by the following transformation of variables:

$$a_{n,m} = \alpha_1^{2m}\tilde{a}_{n,m} \qquad b_{n,m} = \alpha_1^m\left[\tilde{b}_{n,m} + \alpha_0\frac{\alpha_1^m - 1}{\alpha_1 - 1}\right] \qquad \psi_{n,m} = \alpha_1^{-nm}\tilde{\psi}_{n,m} \tag{28}$$

and by an appropriate choice of the coefficients K_m^1 and K_m^0.

2.3. The discrete Volterra lattice

Choosing $f_m^2 = 0$ and $f_m^1(\lambda) = \alpha\lambda$ in equation (8) we obtain

$$\begin{aligned} a_{n,m+1} - a_{n,m} &= \alpha\left(\frac{\pi_{n-1,m+1}}{\pi_{n+1,m}} - \frac{\pi_{n,m+1}}{\pi_{n+2,m}}\right) + \alpha(a_{n,m+1} - a_{n,m}) \\ &\quad + \alpha\frac{\pi_{n,m+1}}{\pi_{n+1,m}}(b_{n,m+1} - b_{n+1,m})\left(b_{n+1,m} + \sum_{j=n+1}^{\infty}(b_{j,m+1} - b_{j+1,m})\right) \end{aligned} \tag{29}$$

$$\begin{aligned} b_{n,m+1} - b_{n,m} &= \alpha\left(\frac{\pi_{n-1,m+1}}{\pi_{n,m}}b_{n-1,m+1} - \frac{\pi_{n,m+1}}{\pi_{n+1,m}}b_{n+1,m}\right) \\ &\quad + \alpha(b_{n,m+1} - b_{n,m}) + \alpha\left(\frac{\pi_{n-1,m+1}}{\pi_{n,m}} - \frac{\pi_{n,m+1}}{\pi_{n+1,m}}\right)\sum_{j=n}^{\infty}(b_{j,m+1} - b_{j+1,m}). \end{aligned} \tag{30}$$

The nonlocal system (29), (30) admits a very natural reduction, $b_{n,m} = 0$ for any n and m. In this case equation (30) is identically satisfied and equation (29) reduces to

$$a_{n,m+1} - a_{n,m} = \frac{\alpha}{1-\alpha}\frac{\pi_{n,m+1}}{\pi_{n+1,m}}(a_{n-1,m+1} - a_{n+1,m}) \tag{31}$$

the discrete–discrete Volterra equation. Equation (31) can also be written as

$$\frac{\pi_{n,m+1}}{\pi_{n+1,m+1}} - \frac{\pi_{n,m}}{\pi_{n+1,m}} + \frac{\alpha}{1-\alpha}\left(\frac{\pi_{n,m+1}}{\pi_{n+2,m}} - \frac{\pi_{n-1,m+1}}{\pi_{n+1,m}}\right) = 0. \tag{32}$$

Setting $a_{n,m} = \sigma v_n(t)$ and $t = m\sigma$ it is easy to show that equation (31) reduces to the Volterra equation

$$\dot{v}_n = \frac{\alpha}{1-\alpha}v_n(v_{n-1} - v_{n+1}) + \mathrm{O}(\sigma). \tag{33}$$

The discrete–discrete Volterra equation (31) corresponds to the following discrete-time evolution of the wavefunction $\psi_{n,m}$:

$$\psi_{n,m+1} = (\alpha - 1)\psi_{n,m} + \alpha\frac{\pi_{n,m+1}}{\pi_{n+2,m}}\psi_{n+2,m}. \tag{34}$$

2.4. The discrete Volterra hierarchy

We can carry out the same reduction which gave us the Volterra equation (31) for the whole discrete-time Toda lattice hierarchy (8). In such a way we obtain

$$a_{n,m+1} - a_{n,m} = g_m^1(\tilde{\mathcal{L}})\left[\frac{\pi_{n-1,m+1}}{\pi_{n+1,m}} - \frac{\pi_{n,m+1}}{\pi_{n+2,m}}\right] + g_m^2(\tilde{\mathcal{L}})[a_{n,m+1} - a_{n,m}] \tag{35}$$

where g_m^1 and g_m^2 are two entire functions of their argument and

$$\tilde{\mathcal{L}}p_{n,m} = a_{n,m+1}\sigma_{n+2,m} - a_{n,m}\sigma_{n,m} + \Sigma_{n-1,m}\frac{\pi_{n-1,m+1}}{\pi_{n+1,m}} - \Sigma_{n+1,m}\frac{\pi_{n,m+1}}{\pi_{n+2,m}} \tag{36}$$

with

$$\sigma_{n+1,m} - \sigma_{n,m} = p_{n-1,m} \qquad \Sigma_{n+1,m} - \Sigma_{n,m} = -p_{n+1,m}\frac{\pi_{n+2,m}}{\pi_{n+1,m+1}}. \tag{37}$$

From equation (35), by choosing $g_m^1 = \tilde{\mathcal{L}}$ and $g_m^2 = 0$, we obtain the higher discrete Volterra equation:

$$\begin{aligned} a_{n,m+1} - a_{n,m} = {} & a_{n,m+1}a_{n+1,m+1} - a_{n,m}a_{n-1,m+1} \\ & + \frac{\pi_{n-1,m+1}}{\pi_{n+1,m}}[a_{n,m} + a_{n-1,m} + a_{n-2,m+1}] + K_{n,m}\left[a_{n,m} + \frac{\pi_{n-1,m+1}}{\pi_{n+1,m}}\right] \\ & - \frac{\pi_{n,m+1}}{\pi_{n+2,m}}[a_{n+2,m} + a_{n+1,m} + a_{n,m+1}] - K_{n+2,m}\left[a_{n,m+1} + \frac{\pi_{n,m+1}}{\pi_{n+2,m}}\right] \end{aligned} \tag{38}$$

where $K_{n,m}$ satisfies the following first-order linear equation:

$$K_{n+1,m} - K_{n,m} = a_{n-1,m} - a_{n-1,m+1}. \tag{39}$$

3. Construction of the Lie point symmetries for the discrete-time Toda lattice

In order to make a complete analysis of the discrete-time Toda lattice (18), let us study its Lie point symmetries, using the algorithm introduced by Levi *et al* [16]. This means that we interpret n and m as two indices characterizing two continuous variables, say x and t, defined on a two-dimensional lattice of points. In the following by $x_{n,m}$ and $t_{n,m}$ we mean the value of the coordinate x and t at the point labelled by the indices n and m. The discrete-time Toda lattice equation ($\Delta_{\text{Toda}} = 0$) is a relation between five points in the two-dimensional plane:

$$\begin{aligned} & P_0 = (x_{n,m}, t_{n,m}) \qquad P_1 = (x_{n+1,m}, t_{n+1,m}) \qquad P_2 = (x_{n,m+1}, t_{n,m+1}) \\ & P_3 = (x_{n-1,m+2}, t_{n-1,m+2}) \qquad P_4 = (x_{n,m+2}, t_{n,m+2}). \end{aligned} \tag{40}$$

On an orthogonal uniform homogeneous lattice the discrete-time Toda lattice can be seen as a relation defining for example π_{P_3}, the field π at the time $m+2$ at the point $n-1$, in terms of points at earlier times for higher values of x

$$\pi_{P_3} = \pi_{P_0} + \pi_{P_2}^2\left(\frac{1}{\pi_{P_1}} - \frac{1}{\pi_{P_4}}\right). \tag{41}$$

Consequently, given the function π on two lines parallel to the t axis, for example those including P_1 and P_0, we can construct the function π for any time at all points at higher values of x (see figure 1).

If the lattice is uniform and homogeneous in both the x and t variables, we can represent the lattice by the following set of equations:

$$x_{n,m} - x_{n+1,m} = x_{n-1,m+2} - x_{n,m+2} \tag{42}$$

$$t_{n,m+1} - t_{n,m} = t_{n,m+2} - t_{n,m+1} \tag{43}$$

$$t_{n,m} = t_{n+1,m} \tag{44}$$

$$x_{n,m} = x_{n,m+1} \tag{45}$$

and their consequences obtained by shifting in n and m. Equations (42)–(45) from now on will be denoted as $\Delta_{\text{lattice}} = 0$. Equations (42)–(45) allow us to cover all points of the plane once two initial points are given.

A Lie point symmetry is defined by giving its infinitesimal generators, i.e. the vector field

$$\hat{X}_P = \xi(P, \pi_P)\partial_x + \tau(P, \pi_P)\partial_t + \phi(P, \pi_P)\partial_{\pi_P} \tag{46}$$

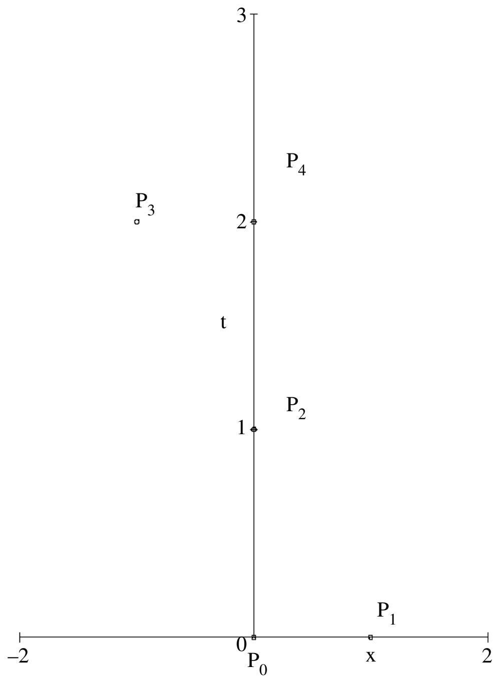

Figure 1. The set of points which are connected by the discrete-time Toda lattice equation on an orthogonal uniform homogeneous lattice.

and generates an infinitesimal transformation in the site P of its coordinates and of the function π_P. The action of (46) on the discrete-time Toda lattice equation (17) is obtained by prolonging (46) to all points of the lattice. The prolongation is obtained [16] by shifting (46) to these points

$$\mathrm{pr}\hat{X} = \hat{X}_{P_0} + \hat{X}_{P_1} + \hat{X}_{P_2} + \hat{X}_{P_3} + \hat{X}_{P_4}. \tag{47}$$

The invariance condition then reads

$$\begin{aligned} &\mathrm{pr}\hat{X}\Delta_{\mathrm{Toda}}|_{(\Delta_{\mathrm{Toda}}=0,\Delta_{\mathrm{lattice}}=0)} = 0 \\ &\mathrm{pr}\hat{X}\Delta_{\mathrm{lattice}}|_{(\Delta_{\mathrm{Toda}}=0,\Delta_{\mathrm{lattice}}=0)} = 0. \end{aligned} \tag{48}$$

The action of (47) on the lattice equation (42) gives

$$\xi(P_0, \pi_{P_0}) - \xi(P_1, \pi_{P_1}) = \xi(P_3, \pi_{P_3}) - \xi(P_4, \pi_{P_4}). \tag{49}$$

Differentiating (49) with respect to π_{P_2} and taking into account the discrete-time Toda lattice equation (41) we obtain $\xi = \xi(P)$ only. Introducing this result into equation (49) and taking into account equations (42)–(45) we obtain

$$\xi(x_{n,m}, t_{n,m}) - \xi(x_{n+1,m}, t_{n,m}) = \xi(2x_{n,m} - x_{n+1,m}, 2t_{n,m+1} - t_{n,m}) - \xi(x_{n,m}, 2t_{n,m+1} - t_{n,m}). \tag{50}$$

Differentiating equation (50) twice, first with respect to $x_{n,m}$ and then to $x_{n+1,m}$, we obtain $\xi = a(t)x + b(t)$. Then, from equation (50) we conclude that $a(t)$ must be a constant. From equation (45), as $t_{n,m}$ and $t_{n,m+1}$ can be taken to be independent variables, we obtain that also $b(t)$ must be a constant and thus

$$\xi_P = a\, x_{n,m} + b. \tag{51}$$

By similar considerations for equations (43) and (44) we obtain that

$$\tau_P = c\, t_{n,m} + d \tag{52}$$

where c and d are arbitrary constants.

Applying the prolonged vector field $\mathrm{pr}\hat{X}$ on the discrete-time Toda lattice equation we obtain

$$\phi(P, \pi_P) = A\,\pi_P \tag{53}$$

where A is an arbitrary constant.

To sum up, the symmetry algebra of the Lie point symmetries of the discrete-time Toda lattice (17) is generated by five elements, given by space and time translations and three independent dilations in the dependent and independent variables.

4. Isospectral and non-isospectral generalized symmetries for the discrete-time Toda lattice

Infinitesimal symmetries for the discrete-time Toda lattice can be obtained as commuting flows, i.e. an infinitesimal symmetry is obtained when its flow in the group parameter and the discrete time evolution commute. These are represented by the hierarchy of nonlinear differential–difference equations associated with the Schrödinger spectral problem (1). We can also consider the nonlinear discrete-time difference equations commuting with the discrete-time Toda lattice, but these turn out not to form a group of symmetry transformations associated with (18). As we shall see later, they provide us with its Bäcklund transformations.

To discuss these issues, it is easier to work in the space of the spectral parameter where the nonlinear evolution of the fields is substituted by the linear evolution of the reflection coefficient. The two spaces are in one to one correspondence for fields which are asymptotically bounded. In such a situation the discrete-time Toda lattice equation is represented in the spectral space by the following evolution of the reflection coefficient $R_m(z, \epsilon)$ (the transmission coefficient $T_m(z, \epsilon)$ is invariant under the m evolution):

$$R_{m+1}(z, \epsilon) = \frac{1 - z\alpha}{1 - \frac{\alpha}{z}} R_m(z, \epsilon) \tag{54}$$

where ϵ is the infinitesimal group parameter (see equation (23)).

Any isospectral deformation ($\frac{\mathrm{d}z}{\mathrm{d}\epsilon_k} = 0$) of the discrete Schrödinger spectral problem (1) is given by

$$\begin{pmatrix} a_{n,m} \\ b_{n,m} \end{pmatrix}_{,\epsilon_k} = (\tilde{\mathcal{L}})^k \begin{pmatrix} a_{n,m}(b_{n,m} - b_{n+1,m}) \\ a_{n-1,m} - a_{n,m} \end{pmatrix}. \tag{55}$$

The recursion operator $\tilde{\mathcal{L}}$ is

$$\tilde{\mathcal{L}} \begin{pmatrix} p_{n,m} \\ q_{n,m} \end{pmatrix} = \begin{pmatrix} p_{n,m} b_{n+1,m} + a_{n,m}(q_{n,m} + q_{n+1,m}) + (b_{n,m} - b_{n+1,m}) s_{n,m} \\ b_{n,m} q_{n,m} + p_{n,m} + s_{n-1,m} - s_{n,m} \end{pmatrix} \tag{56}$$

with $s_{n,m}$ given by an asymptotically bounded solution of the inhomogeneous first-order equation

$$s_{n+1,m} = \frac{a_{n+1,m}}{a_{n,m}} (s_{n,m} - p_{n,m}). \tag{57}$$

The index k of ϵ_k denotes the fact that this group parameter is associated with the kth equation of the Toda lattice hierarchy (55). In correspondence with equation (55) we have an evolution (in ϵ_k) of the reflection coefficient associated with the discrete Schrödinger spectral problem (1), i.e.

$$\frac{\mathrm{d}R_m(z, \epsilon_k)}{\mathrm{d}\epsilon_k} = \mu\lambda^k R_m(z, \epsilon_k) \tag{58}$$

with

$$\lambda = z + \frac{1}{z} \qquad \mu = \frac{1}{z} - z. \tag{59}$$

It is easy to prove that the flows (14) and (55) commute by checking that the corresponding flows of the reflection coefficients, given by equations (54) and (58), commute.

A less obvious calculation has to be performed to obtain the non-isospectral symmetries of the discrete-time Toda lattice equation. In this case we have

$$\begin{pmatrix} a_{n,m} \\ b_{n,m} \end{pmatrix}_{,\epsilon_k} = f_m^k(\tilde{\mathcal{L}}) \begin{pmatrix} a_{n,m}(b_{n,m} - b_{n+1,m}) \\ a_{n-1,m} - a_{n,m} \end{pmatrix} + \tilde{\mathcal{L}}^k \begin{pmatrix} a_{n,m}[(2n+3)b_{n+1,m} - (2n-1)b_{n,m}] \\ b_{n,m}^2 - 4 + 2[(n+1)a_{n,m} - (n-1)a_{n-1,m}] \end{pmatrix}. \tag{60}$$

The function $f_m^k(\lambda)$ depends on the equation under consideration and, for the discrete-time Toda lattice, is obtained as a solution of the difference equation:

$$f_{m+1}^k(\lambda) - f_m^k(\lambda) = -2\lambda^k \frac{2\alpha^2 - \alpha\lambda}{1 + \alpha^2 - \alpha\lambda}. \tag{61}$$

Up to an arbitrary inessential constant the function $f_m^k(\lambda)$ is given by

$$f_m^k(\lambda) = -2m\lambda^k \frac{2\alpha^2 - \alpha\lambda}{1 + \alpha^2 - \alpha\lambda}. \tag{62}$$

The proof that the flow (60) with f_m^k given by (62) commutes with that of equation (14) is easily obtained in the space of the spectrum, where the reflection coefficient associated with equation (60) satisfies the equation

$$\frac{\mathrm{d}R_m(z, \epsilon_k)}{\mathrm{d}\epsilon_k} = \mu f_m^k(\lambda) R_m(z, \epsilon_k) \qquad \lambda_{\epsilon_k} = \mu^2 \lambda^k. \tag{63}$$

On the lhs we have the total derivative of $R_m(z, \epsilon_k)$ with respect to ϵ_k.

Both the isospectral (for $k \neq 0$) and non-isospectral symmetries involve the dependent variable at different points of the lattice and, even if the continuous limit will correspond to Lie point symmetries [17], they are effectively generalized symmetries. As such they are not integrable, i.e. we are not able to use them to obtain group transformations. They can be used to provide solutions via symmetry reduction. As an example of these symmetries we write down the simplest non-isospectral symmetry obtained for $k = 0$ and $\alpha = 1$ and given by

$$\begin{pmatrix} a_{n,m} \\ b_{n,m} \end{pmatrix}_{,\epsilon_0} = -2m \begin{pmatrix} a_{n,m}(b_{n,m} - b_{n+1,m}) \\ a_{n-1,m} - a_{n,m} \end{pmatrix} + \begin{pmatrix} a_{n,m}[(2n+3)b_{n+1,m} - (2n-1)b_{n,m}] \\ b_{n,m}^2 - 4 + 2[(n+1)a_{n,m} - (n-1)a_{n-1,m}] \end{pmatrix}. \tag{64}$$

Taking into account equation (10), we can rewrite equation (64) as

$$\begin{aligned} (\pi_{n,m})_{,\epsilon_0} &= \pi_{n,m}\left\{ -(2m + 2n + 1)b_{n,m} + 2\sum_{j=n}^{\infty} b_{j,m} \right\} \\ (b_{n,m})_{,\epsilon_0} &= b_{n,m}^2 - 4 + 2[(n + m + 1)a_{n,m} - (n + m - 1)a_{n-1,m}]. \end{aligned} \tag{65}$$

In view of equation (16), $b_{n,m}$ can be rewritten in terms of $\pi_{n,m}$ and its shifted values.

A symmetry reduction with respect to the symmetry given by equation (65) is obtained by solving the discrete-time Toda lattice (17) together with the equation we obtain by equating to zero the rhs of equation (65), i.e.

$$\begin{aligned} &(2m + 2n - 1)b_{n,m} - (2m + 2n + 3)b_{n+1,m} = 0 \\ &a_{n,m}[2(n+1) + 2m] - a_{n-1,m}[2(n-1) + 2m] = 4 - b_{n,m}^2. \end{aligned} \tag{66}$$

The general solution is given by

$$b_{n,m} = \frac{b_m^0}{(2m+2n-1)(2m+2n+1)}$$
$$a_{n,m} = \frac{1}{(2n+2m+2)(2n+2m)}\left[a_m^0 + 4n(2m+1+n) + \frac{(b_m^0)^2}{4(2m+2n+1)^2}\right]. \tag{67}$$

Using equations (14) and (15) with $\alpha = 1$, we obtain two equations for b_m^0 and a_m^0, the *reduced equations*. The study of all the possible reductions of the discrete-time Toda lattice with respect to the Lie point and generalized symmetries is out of the scope of this presentation.

5. Bäcklund transformations and symmetries

Bäcklund transformations are obtained by the same kind of formula as used to obtain the difference–difference equations when the new functions $(\tilde{a}_{n,m}, \tilde{b}_{n,m})$ are defined as

$$\tilde{a}_{n,m} = a_{n,m+1} \qquad \tilde{b}_{n,m} = b_{n,m+1}. \tag{68}$$

With this identification the class of Bäcklund transformations associated with the discrete-time Toda lattice hierarchy reads

$$\delta(\Lambda)\begin{pmatrix} (\tilde{b}_{n,m} - b_{n+1,m})\dfrac{\tilde{\pi}_{n,m}}{\pi_{n+1,m}} \\ \dfrac{\tilde{\pi}_{n-1,m}}{\pi_{n,m}} - \dfrac{\tilde{\pi}_{n,m}}{\pi_{n+1,m}} \end{pmatrix} = \gamma(\Lambda)\begin{pmatrix} \tilde{a}_{n,m} - a_{n,m} \\ \tilde{b}_{n,m} - b_{n,m} \end{pmatrix} \tag{69}$$

where Λ is the Bäcklund recursion operator, obtained in the same way as $\mathcal{L}$, and given by

$$\Lambda\begin{pmatrix} p_{n,m} \\ q_{n,m} \end{pmatrix} = \begin{pmatrix} \tilde{a}_{n,m}(q_{n,m} + q_{n+1,m}) + (a_{n,m} - \tilde{a}_{n,m})\tilde{P}_{n,m} \\ p_{n,m} + \tilde{\Sigma}_{n-1,m} - \tilde{\Sigma}_{n,m} + \tilde{b}_{n,m}q_{n,m} \end{pmatrix}$$
$$+ \begin{pmatrix} b_{n+1,m}p_{n,m} + (\tilde{b}_{n,m} - b_{n+1,m})\tilde{\Sigma}_{n,m} \\ (b_{n,m} - \tilde{b}_{n,m})\tilde{P}_{n,m} \end{pmatrix}. \tag{70}$$

Above, $\tilde{\Sigma}_{n,m}$ and $\tilde{P}_{n,m}$ are now defined as the bounded solutions to the following difference equations:

$$\tilde{P}_{n,m} - \tilde{P}_{n+1,m} = q_{n,m}$$
$$\tilde{\Sigma}_{n,m}\frac{\pi_{n+1,m}}{\tilde{\pi}_{n,m}} - \tilde{\Sigma}_{n+1,m}\frac{\pi_{n+2,m}}{\tilde{\pi}_{n+1,m}} = p_{n,m}\frac{\pi_{n+1,m}}{\tilde{\pi}_{n,m}}. \tag{71}$$

γ and δ are entire functions of their arguments. Equation (70) corresponds asymptotically to

$$\tilde{R}_m = \frac{\gamma(\lambda) - z\delta(\lambda)}{\gamma(\lambda) - \frac{\delta(\lambda)}{z}} R_m. \tag{72}$$

The simplest Bäcklund transformation is obtained by choosing $\gamma = 1$ and δ constant and reads

$$\tilde{a}_{n,m} - a_{n,m} = \delta(\tilde{b}_{n,m} - b_{n+1,m})\frac{\tilde{\pi}_{n,m}}{\pi_{n+1,m}}$$
$$\tilde{b}_{n,m} - b_{n,m} = \delta\left[\frac{\tilde{\pi}_{n-1,m}}{\pi_{n,m}} - \frac{\tilde{\pi}_{n,m}}{\pi_{n+1,m}}\right]. \tag{73}$$

It is worthwhile to recall that while the composition of two Bäcklund transformations is still a Bäcklund transformation, however of higher order, the Bäcklund transformations do not form a Lie group as the product of two Bäcklund transformations does not give a Bäcklund transformation of the same form as the original ones. Moreover, the theorems presented in [18] for the Toda lattice equation are also valid in this case, i.e. any Bäcklund transformation can be written as a superposition of an infinite number of symmetries and vice versa.

6. Conclusions

We have shown that we are able to construct hierarchies of integrable difference–difference equations. In particular we have constructed the discrete hierarchy associated with the discrete Schrödinger spectral problem. Using the recursion operator we have constructed the discrete-time Toda lattice, the discrete-time Volterra equation and a higher member of its hierarchy. Moreover we considered the inhomogeneous Toda lattice.

For these equations one can construct both symmetries and Bäcklund transformations. Here, for the sake of simplicity, we limit ourselves to the case of the discrete-time Toda lattice only. The infinitesimal Lie symmetries of the discrete-time Toda lattice equation are given by differential–difference equations commuting with it. The isospectral symmetries are represented by the nonlinear differential–difference equations of the Toda lattice hierarchy. In the non-isospectral case, the symmetries are a non-isospectral extension giving Toda lattice equations with non-constant coefficients.

As the equations are already discrete in all variables, the Bäcklund transformations do not provide any new information and are given by the equations themselves. They are flows commuting with the equations but do not have the properties of a Lie group.

Acknowledgments

This work has been carried out with the financial support of INTAS 99-1782 and the Cofin project 'Singolaritá, Integrabilitá e Simmetrie'. We thank Pavel Winternitz for helpful discussions.

References

[1] Calogero F and Degasperis A 1982 *Spectral Transform and Solitons: Tools to Solve and Investigate Nonlinear Evolution Equations* (Amsterdam: North-Holland)

[2] Bruschi M, Ragnisco O and Levi D 1981 Evolution equations associated to the discrete analog of the matrix Schrödinger spectral problem solvable by the inverse spectral transform *J. Math. Phys.* **22** 2463–71

[3] Bruschi M and Ragnisco O 1981 Nonlinear differential–difference equations, associated Bäcklund transformations and Lax technique *J. Phys. A: Math. Gen.* **14** 1075–81

[4] Ablowitz M J and Ladik J F 1976 A nonlinear difference scheme and inverse scattering *Stud. Appl. Math.* **55** 213–29

[5] Hirota R 1977 Nonlinear partial difference equations. I. A difference analogue of the Korteweg–de Vries equation *J. Phys. Soc. Japan* **43** 1424–33

[6] Hirota R 1977 Nonlinear partial difference equations. II. Discrete-time Toda equation *J. Phys. Soc. Japan* **43** 2074–8

[7] Hirota R 1977 Nonlinear partial difference equations. III. Discrete sine–Gordon equation *J. Phys. Soc. Japan* **43** 2079–86

[8] Hirota R 1978 Nonlinear partial difference equations. IV. Bäcklund transformations for the discrete-time Toda equation *J. Phys. Soc. Japan* **45** 321

[9] Orfanidis S J 1978 Discrete sine–Gordon equations *Phys. Rev.* D **18** 3822

Levi D, Pilloni L and Santini P M 1981 Integrable three-dimensional lattices *J. Phys. A: Math. Gen.* **14** 1567–75

[10] Nijhoff F W, Papageorgiou V G and Capel H W 1992 Integrable time-discrete systems: lattices and mappings. *Quantum groups, Proc. Workshops, Euler Int. Math. Inst. Leningrad/USSR 1990 (Lecture Notes in Mathematics Vol. 1510)* pp 312–25

[11] Maeda S 1980 Canonical structure and symmetries for discrete systems *Math. Japan.* **25** 405–20

[12] Quispel G R W, Capel H W and Sahadevan R 1992 Continuous symmetries of differential–difference equations: the Kac–van Moerbeke equation and Painlevé reduction *Phys. Lett.* A **170** 379–83

[13] Quispel G R W and Sahadevan R 1993 Lie symmetries and the integration of difference equations *Phys. Lett.* A **184** 64–70

[14] Byrnes G B, Sahadevan R and Quispel G R W 1995 Factorizable Lie symmetries and the linearization of difference equations *Nonlinearity* **8** 443–59

[15] Sahadevan R and Quispel G R W 1997 Lie symmetries and linearisation of the QRT mapping *Physica* A **234** 775–84

[16] Levi D, Tremblay S and Winternitz P Point symmetries of multi-dimensional nonlinear partial difference equations *J. Phys. A: Math. Gen.* at press

[17] Hernandez Heredero R, Levi D, Rodriguez M A and Winternitz P 2000 Lie algebra contractions and symmetries of the Toda hierarchy *J. Phys. A: Math. Gen.* **33** 5025–40

[18] Hernandez Heredero R, Levi D, Rodriguez M A and Winternitz P 2001 Relation between Bäcklund transformations and higher continuous symmetries of the Toda equation *J. Phys. A: Math. Gen.* **34** 2459–65

[19] Levi D and Ragnisco O 1991 The inhomogeneous Toda lattice: its hierarchy and Darboux–Bäcklund transformations *J. Phys. A: Math. Gen.* **24** 1729–39

Levi D 1992 Inhomogeneous Toda lattices and their symmetries *Group Theory in Physics (Cocoyoc, 1991) (AIP Conf. Proc. Vol. 266)* (New York: AIP) pp 97–108